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A MODEL OF NODAL ENTROPY IN A TRANSPORTATION NETWORK
WITH CONGESTION COSTS. / Scott, Allen J. 1971.

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A Model of Nodal Entropy in a Transportation Network with Congestion Costs*

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An elementary model defining maximum entropy populations for a set of nodes is developed. These nodes are connected by a simple transportation network to a central point where all work places are concentrated. A congestion cost function is defined for network arcs. Then the model yields an equilibrium solution that identifies nodal populations as an entropic function of the total cost of the journey to work.

In this paper an attempt is made to construct a theoretical model of nodal entropy in a simple transportation network, where the cost of transportation along the network is augmented by a congestion factor. In the present context, the term *nodal entropy* signifies a system-wide measure of the general distribution of population over the nodes of the network. Moreover, maximization of this entropy measure yields a criterion of equilibrium for the entire set of nodal populations. This criterion of equilibrium is based on the explicit assumption that any population distribution is uniquely determined by a transport cost principle. Then, the optimization of the nodal entropy measure yields, first, a set of maximum likelihood populations for the nodes of the system, and, second, a derived set of congestion costs for the arcs of the system. Moreover, both the set of populations and the set of congestion costs are such as to be in a state of secondary equilibrium with respect to each other.

The present analysis is derived out of papers by WILSON^[5] and SCOTT,^[3]

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and the following description takes these two papers as its point of departure.

DESCRIPTION OF A SIMPLE NODE-NETWORK SYSTEM

SUPPOSE THERE exists some geographic region with total population given as T households. This population is distributed in some manner over a set of n spatially dispersed nodes. Any node, i , has a total of x_i households. For simplicity, assume that only one person in each household goes out to work. Thus, the term x_i also defines the number of workers who reside at node i . Let it be further assumed that all work places are concentrated at a single central point. This assumption can be fairly easily relaxed, though it is maintained here for simplicity of notation and analysis.

Each worker must now journey daily from his place of residence to the central work point, and the geographical properties of this journey to work are related to and governed by an underlying transportation network. Let this network be of uniform carrying capacity throughout. In addition, in the present study, the graph of this network is rigidly restricted to the form of a tree. This means that there is only ever one possible route that a worker can take from his home node to the central work point. This characteristic in turn enables the present analysis to skirt around the difficult and largely intractable problems that would arise if alternative journey to work routes were allowed to develop as congestion costs come into being along shortest journey to work routes. These congestion costs are here made to be directly a function of network flow, and they represent incremental costs that are incurred in addition to the ordinary direct cost of travel.

The problem of how to define an equilibrium distribution of households over the nodes of this system now presents itself, and in such a way that this distribution is in some sense in harmony with the various costs associated with the transportation process. As has already been suggested, this is to be accomplished in the analysis which follows by means of an entropy maximizing principle. This principle is underpinned by the consideration that system-wide expenditures on the journey to work will always sum to the constant quantity, C . This assumption may no doubt at first appear to be somewhat inflexible. However, Wilson^[5] has argued in favor of this assumption by invoking an analogy with the notion of physical entropy as treated in statistical mechanics; and Scott^[3] has suggested that the assumption can be sustained by relating it to certain principles of consumer indifference analysis. Thus, the assumption will be accepted here without further discussion. However, it may in addition be noted that TOMLIN AND TOMLIN^[4] in a study of simple traffic flows have suggested an alternative formulation of this cost condition that in practice seems to relate to average rather than total expenditures on transportation.

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The model that is now to be developed takes as its basic ingredients the elements described above and molds these into a maximum entropy system. The outputs of this model are the populations of the different network nodes and the congestion costs along all network arcs.

A MATHEMATICAL MODEL

GIVEN THAT THE variable x_i is the number of workers (or households) residing at node i , then

$$L = - \sum_{i=1}^{i=n} x_i \log x_i \quad (1)$$

is the global measure of entropy for the total system of n nodes. This measure is derived (via Stirling's approximation) from a general combinatorial expression of the entropy principle. This expression is $T! / (x_1!)(x_2!) \cdots (x_n!)$ and it identifies the total number of different ways in which T elements (here, workers) can be distributed over the set $\{x_1, x_2, \dots, x_n\}$. Then, optimization of this combinatorial expression, or of the derivative function (1), yields numerical values for the set $\{x_1, x_2, \dots, x_n\}$ which are maximally likely in a statistical sense. The detailed derivation and interpretation of the function (1) have been discussed by Wilson,^[5] and reference should be made to Wilson's work for a more complete treatment of these matters.

In addition, in the present instance, the function (1) is subjected to two conditions. The first condition is that the sum of all x_i must be equal to the constant value T . The second condition is that all expenditures on the journey to work must be equal to C . Recall that these expenditures on the journey to work include both an ordinary (uncongested) transportation cost and an incremental cost due directly to congestion. The exact nature of these costs is at this point considered explicitly.

Congested and Uncongested Transportation Costs

Let the arc i be that arc in the transportation tree that leads directly out of node i in the direction of the central work point. In addition, let the set $A(i)$ designate all those arcs that converge directly and indirectly upon node i . That is, the set $A(i)$ includes every arc that is so located in the transportation network that a journey along that arc to the central point must ultimately pass through node i . Thus, in Fig. 1, for example, the arcs 6 and 9 are elements of the set $A(1)$, while the arc 9 is the sole element of the set $A(6)$. It is now readily apparent that total flow along the i th arc can be written

$$x_i^* = x_i + \sum_{j \in A(i)} x_j. \quad (2)$$

Now let the cost of *uncongested* travel per person along the arc i be denoted c_i . By contrast, the total *congested* cost of travel per unit of distance along

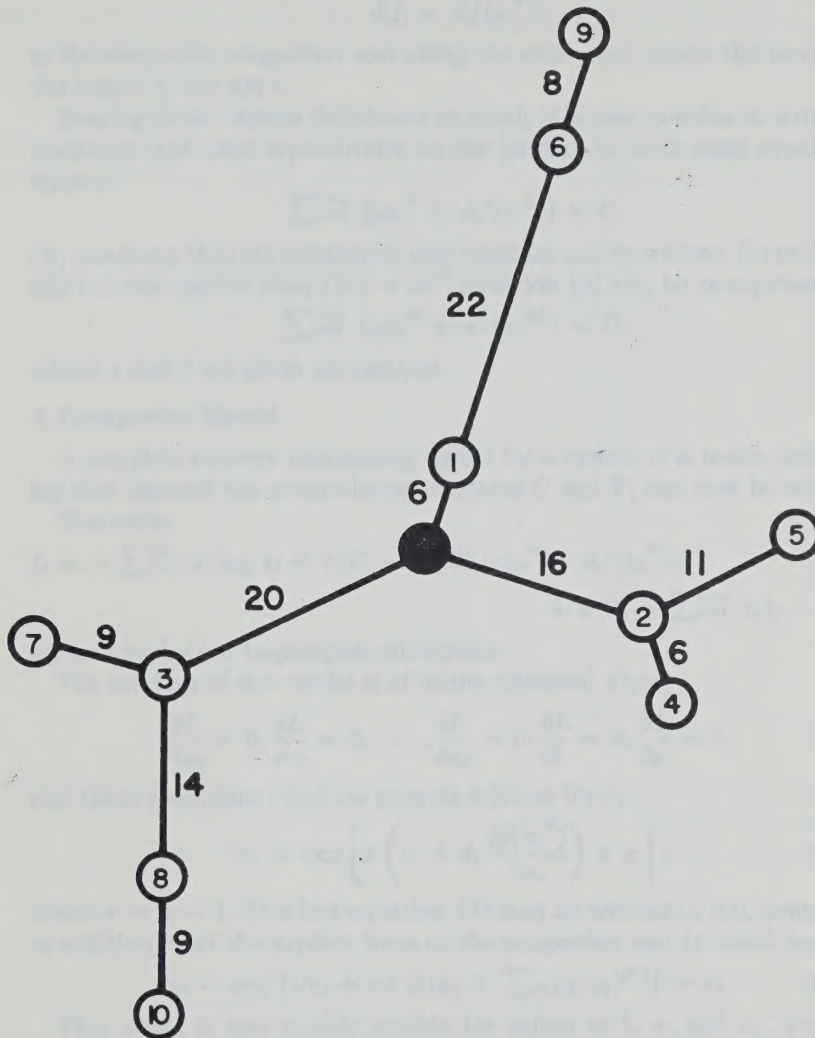


Fig. 1. Simple transportation network with residential nodes and central work point.

the arc i is

$$f_i = f(x_i^*), \quad (3)$$

so that the congestion costs are simply directly a function of total flow. Observe that since all arcs are taken to be of uniform capacity, there is therefore no need to enter an arc-capacity term into this congestion cost function.

Now, if (3) is the congestion cost per unit of distance along the arc i , then

$$d_i f_i = d_i f(x_i^*) \quad (4)$$

is the composite congestion cost along the entire arc, where the term d_i is the length of the arc i .

Bearing these various definitions in mind, it is now possible to write the condition that total expenditures on the journey to work must equal C as follows:

$$\sum_{i=1}^{i=n} [c_i x_i^* + d_i f(x_i^*)] = C. \quad (5)$$

Or, assuming that the congestion cost function can be written (hypothetically) in the explicit form $f(x) = \alpha x^\beta$, equation (5) can be re-expressed as,

$$\sum_{i=1}^{i=n} (c_i x_i^* + \alpha d_i x_i^{*\beta}) = C, \quad (6)$$

where α and β are given parameters.

A Composite Model

A complete entropy maximizing model for a system of n nodes, and taking into account the constraining constants C and T , can now be written:

Maximize:

$$L = - \sum_{i=1}^{i=n} x_i \log x_i + \lambda \{C - \sum_{i=1}^{i=n} [c_i x_i^* + d_i f(x_i^*)]\} + \mu (T - \sum_{i=1}^{i=n} x_i), \quad (7)$$

where λ and μ are Lagrangian multipliers.

The solution of this model is of course obtained where

$$\frac{\partial L}{\partial x_1} = 0, \frac{\partial L}{\partial x_2} = 0, \dots, \frac{\partial L}{\partial x_n} = 0, \frac{\partial L}{\partial \lambda} = 0, \frac{\partial L}{\partial \mu} = 0, \quad (8)$$

and these conditions yield the general solution for x_i

$$x_i = \exp \left[\lambda \left(c_i + d_i \frac{\partial f(x_i^*)}{\partial x_i} \right) + \pi \right], \quad (9)$$

where $\pi = \mu - 1$. This last equation (9) may be written in full, assuming, in addition, that the explicit form of the congestion cost function applies:

$$x_i = \exp \{ \lambda [c_i + \alpha \beta d_i (x_i + \sum_{j \in A(i)} x_j)^{\beta-1}] + \pi \}. \quad (10)$$

This model is now readily soluble for values of λ , π , and x_i . For the purposes of numerical resolution of this model, Newton's iterative algorithm for the solution of a set of nonlinear simultaneous equations has been found to give excellent results. A general description of this algorithm and of its use in the numerical analysis of entropy maximizing models is given by Scott.^[3]

A NUMERICAL EXAMPLE

AN EXEMPLARY numerical problem is now considered, and the elements of this problem are displayed graphically in Fig. 1. The system represented

by Fig. 1 consists of a central point in which all places of work are concentrated, and, surrounding this point, a set of ten nodes that collectively contain the T households. The transportation network for this system is shown in Fig. 1 as a set of arcs. The distance along each arc is clearly indicated in Fig. 1.

In this sample problem it is assumed for simplicity that $c_i = d_i$. The congestion cost parameters α and β are fixed at 0.05 and 1.50 respectively. In addition, the total number of households, T , is held constant at 100. Then

TABLE I
Maximum Entropy Solution with $C = 1000$

Node (i)	Households	Total Congestion Costs on Arc i
1	90.0	255.8
2	5.7	17.3
3	2.3	3.5
4	2.0	0.9
5	.0	.0
6	.0	.0
7	.0	.0
8	.0	.0
9	.0	.0
10	.0	.0
Total	100.0	277.5
$\lambda = -0.30$		$\pi = 7.62$

given these parameters, a set of maximum entropy solutions were computed for this system for a succession of different values of C . These values are 1000, 2000, 3000. The results of these computations are shown in Tables I, II, and III.

The tables clearly demonstrate that as total expenditures on transportation increase, households become more and more widely scattered over the entire network system. Depending on the configuration of the network, this in some cases alleviates the amount of congestion on network arcs, and in other cases it increases the amount of congestion. Nevertheless, over-all congestion costs tend very clearly to grow larger as total expenditures on transportation increase. This growth in total congestion costs probably results from increases in the length of individual work trips as much as from increases in load on any specific set of arcs. Finally, note that the spatial distribution of household densities is approximately identified as being in a negative exponential relation to distance from the central point; and indeed when all congestion costs are zero this negative exponential relation is seen to hold without qualification. This purely theoretical ob-

servation has of course been widely corroborated in a large number of empirical studies of urban population densities (c.f., CASETTI^[2]). In addition, BUSSIÈRE AND SNICKARS^[1] have recently provided an alternative derivation of this negative exponential rule from entropy maximizing principles.

TABLE II
Maximum Entropy Solution with $C = 2000$

Node (i)	Households	Total Congestion Costs on Arc i
1	44.0	111.2
2	12.2	129.6
3	8.3	80.7
4	11.1	11.1
5	6.5	9.0
6	4.6	23.2
7	5.6	5.9
8	3.2	7.3
9	3.0	2.1
10	1.6	0.9
Total	100.0	381.1
$\lambda = -0.10$		$\pi = 4.68$

TABLE III
Maximum Entropy Solution with $C = 3000$

Node (i)	Households	Total Congestion Costs on Arc i
1	15.2	55.4
2	11.4	147.7
3	10.0	208.2
4	11.2	11.2
5	9.9	17.1
6	8.9	78.7
7	9.6	13.4
8	8.3	42.8
9	8.4	9.7
10	7.2	8.8
Total	100.0	592.5
$\lambda = -0.02$		$\pi = 2.90$

However, in the model considered here the differential effects of the costs of congestion make this negative exponential relation somewhat less than perfect. It is likely that this phenomenon of congestion may provide at least a partial explanation of some of the discrepancies that exist in empirical

cases between the negative exponential function and observed distributions of population in urban areas. Providing that congestion cost data are available, this hypothesis may easily be empirically tested by means of the model formalized in equation (10).

CONCLUSION

A MODEL OF nodal entropy in a transportation network has been derived and demonstrated. The model yields as a primary solution the distribution of population around a focal point. This distribution is largely a response to the differential effects of the cost of transportation through the network leading to the central point. A secondary output of the model is various congestion costs in different parts of the network.

The enrichment of the model presented above by extension of certain of its basic assumptions would at this point probably greatly repay the effort. Three main directions of development of the model would seem to be especially worthwhile in view of the work accomplished above. First, to extend the model so that it is able to handle transport networks of any degree of complexity, and not simply networks that are in the form of a tree. Second, to relate the cost of congestion directly to arc capacity, and to permit capacity to vary from arc to arc. Third, to develop investment functions for the transportation network so as to yield rational investment criteria in relation to the distribution of population and the cost of congestion. Each of these extensions would greatly contribute to further understanding of some of the most important processes underlying the development of node-network systems as entropic and organic entities.

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